A cylindrical sound pulse in a rotating gas

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This paper deals with the propagation of a sound pulse into a gas which initially has solid-body rotation and constant temperature, the initial pressure and density increasing outwards like e^x, where x is the square of a certain dimensionless radial co-ordinate. The perturbations are due to a source-like disturbance on the axis of symmetry, which begins to act at time $t = 0$: most attention is paid to source strengths which vary in time like a Dirac pulse or a step function, but the following remarks apply generally.

Immediately behind the wave front the perturbation velocity and temperature decay like $e^{-\frac{1}{2}x}$, while the (absolute) perturbation pressure and density grow like $e^{\frac{1}{2}x}$ (the relative pressure and density increments, which are referred to local conditions in the undisturbed state, then also decay like $e^{-\frac{1}{2}x}$). The rotation also introduces oscillations in flows which, with the same disturbance at the origin and no rotation, would vary monotonically with time at a given point.

1. Introduction

The problem studied in this paper is the following. An infinite mass of perfect gas rotates with constant angular velocity about an axis of symmetry and is at constant temperature: the pressure and density increase radially. At time $t = 0$ a weak release of mass or of energy begins at the axis, or a small cylinder begins to expand there, so that cylindrical sound waves radiate outwards. This problem seems worth while for two reasons. First, interest has recently revived in all flows of a fluid with solid-body rotation: these are simple examples of flows in which vorticity plays an important role. Secondly, the problem involves the study of sound waves propagating into a variable medium.

The rotation introduces two principal effects. On a linearized theory the Coriolis force restrains radial displacements of a fluid particle in precisely the same way that a spring restrains the motion of a solid particle, and this effect would be expected to give the flow an oscillatory character. In incompressible flows of a rotating fluid with an axial velocity (see, for example, Squire 1956) this oscillatory behaviour is very evident. In addition, the variable pressure of the undisturbed medium introduces a damping effect which dominates conditions immediately behind the wave front.

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2. Formulation

Let (r, θ) be polar co-ordinates, (u, v) the corresponding components of velocity, (p, ρ, T) the pressure, density and temperature, c the sound speed, and γ the specific-heat ratio of a perfect gas. Conditions in the undisturbed state are denoted by suffix $(\,)_0$, and perturbations by suffix $(\,)_1$.

For the undisturbed state we take steady, solid-body rotation and constant temperature:

$$u_0 = 0, \quad v_0 = \Omega r, \quad T_0 = \text{constant}. \tag{2.1}$$

The radial momentum equation becomes

$$\frac{(\Omega r)^2}{r} = \frac{1}{\rho_0} p_{0r} = \frac{c_0^2}{\gamma} \frac{p_{0r}}{p_0},$$

where lettered suffices denote partial differentiation; hence

$$p_0(r) = p_0(0) \exp\left(\frac{\gamma \Omega^2 r^2}{2c_0^2}\right), \quad \rho_0(r) = \frac{p_0(0)\,\gamma}{c_0^2} \exp\left(\frac{\gamma \Omega^2 r^2}{2c_0^2}\right). \tag{2.2}$$

The rates of strain and viscous stresses all vanish, and there is no dissipation, no heat conduction, and no change of entropy along a streamline, so that this basic flow is an exact solution of the Navier–Stokes equations. The basic state is presumably that which would result in the laboratory from the prolonged spinning of a cylindrical container of gas.

Henceforth we neglect viscosity and heat conduction. The disturbances are to be functions of (r, t) only, and there is no velocity parallel to the axis. In writing the equations of motion, we permit the existence of weak distributed mass and heat sources of strength $Q^{(m)}$ and $Q^{(h)}$, respectively. Then

$$(r\rho)_t + (r\rho u)_r = rQ^{(m)}, \tag{2.3}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right) u - \frac{v^2}{r} = -\frac{1}{\rho} p_r, \tag{2.4}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right) (vr) = 0, \tag{2.5}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right) \frac{p}{\rho^\gamma} = \frac{\gamma - 1}{\rho^\gamma} Q^{(h)}. \tag{2.6}$$

Writing $p = p_0(r) + p_1(r, t)$, etc., and linearizing the disturbance terms, we obtain

$$(r\rho_1)_t + (r\rho_0 u_1)_r = rQ^{(m)}, \tag{2.7}$$

$$u_{1t} - 2\Omega v_1 = -\frac{1}{\rho_0} p_{1r} + \frac{\rho_1}{\rho_0^2} p_{0r}, \tag{2.8}$$

$$v_{1t} + 2\Omega u_1 = 0, \tag{2.9}$$

$$p_{1t} - c_0^2 \rho_{1t} - (\gamma - 1) p_{0r} u_1 = (\gamma - 1) Q^{(h)}. \tag{2.10}$$

If ξ is a radial displacement, such that $\xi_t = u_1$ to first order, the momentum equations (2.8) and (2.9) may be written

$$\xi_{tt} + 4\Omega^2 \xi = -\frac{1}{\rho_0} p_{1r} + \frac{\rho_1}{\rho_0^2} p_{0r},$$

which shows the spring-like effect of the Coriolis terms.

It is convenient to introduce $r\rho_0 u_1 = q$, say, in place of u_1. The function q is $1/2\pi$ times the mass flow across a circle $r=$ constant and will be called the 'outflow'. Cross-differentiating (2.8) and (2.10) to eliminate p_1, and substituting for ρ_{1t} from (2.7) and for v_{1t} from (2.9), we obtain

$$q_{rr}-\frac{1}{r}q_r-\frac{1}{c_0^2}q_{tt}-\frac{\gamma\Omega^2}{c_0^2}rq_r-\frac{4\Omega^2}{c_0^2}q = rQ_r^{(m)}-\frac{\Omega^2r^2}{c_0^2}Q^{(m)}+\frac{\gamma-1}{c_0^2}rQ_r^{(h)}. \tag{2.11}$$

We have retained source terms only to show that mass and heat sources on the axis make similar contributions: hereafter we consider only sources on the axis, and incorporate their presence in boundary conditions; the equations will be written for source-free flow.

In terms of $u_1(=u)$ equation (2.11) is then

$$u_{rr}+\frac{1}{r}u_r-\frac{1}{r^2}u-\frac{1}{c_0^2}u_{tt}+\frac{\gamma\Omega^2}{c_0^2}ru_r-(4-\gamma)\frac{\Omega^2}{c_0^2}u = 0. \tag{2.12}$$

Alternatively, we may introduce a stream function ψ such that

$$\psi_t = r\rho_0 u_1 = q, \quad \psi_r = -r\rho_1, \tag{2.13}$$

and such that $\psi = 0$ ahead of the wave front bounding the disturbed region; ψ is then $1/2\pi$ times the perturbation mass between infinity and a circle $r=$ constant. If (2.11) is integrated with respect to time, the arbitrary function of r which occurs is zero in view of conditions ahead of the wave front, and there results

$$\psi_{rr}-\frac{1}{r}\psi_r-\frac{1}{c_0^2}\psi_{tt}-\frac{\gamma\Omega^2}{c_0^2}r\psi_r-\frac{4\Omega^2}{c_0^2}\psi = 0. \tag{2.14}$$

u_1 and ρ_1 are given in terms of ψ by (2.13), and

$$\left.\begin{aligned} v_1 &= -\frac{2\Omega}{r\rho_0}\psi, \\ p_1 &= (\gamma-1)\,\Omega^2\psi-c_0^2\frac{\psi_r}{r}, \\ c_1 &= \frac{\gamma-1}{2}\frac{c_0}{p_0}\left(\Omega^2\psi-\frac{c_0^2}{\gamma}\frac{\psi_r}{r}\right). \end{aligned}\right\} \tag{2.15}$$

Equations (2.12) and (2.14) involve the wave equation operator together with a damping term (of opposite sign in the two equations) and a stiffness term.

3. The propagation of weak shocks and expansion waves

The importance of the damping term immediately behind a wave front can be seen rather simply. We introduce characteristic variables $c_0t+r=\alpha$, $c_0t-r=\beta$, and write $\gamma\Omega^2/2c_0^2 = k$; then (2.14) becomes

$$4\psi_{\alpha\beta}+\left\{\frac{2}{\alpha-\beta}+k(\alpha-\beta)\right\}(\psi_\alpha-\psi_\beta)+\frac{4\Omega^2}{c_0^2}\psi = 0. \tag{3.1}$$

If now ψ_β is discontinuous across $\beta = b$, say, while ψ, the perturbation mass, is continuous, we write

$$[\psi_\beta]_{b-}^{b+} = \frac{1}{2}\left[\frac{r\rho_0u_1}{c_0}+r\rho_1\right]_{b-}^{b+} = j(\alpha),$$

and obtain from (3.1)

$$4j_\alpha - \left\{\frac{2}{\alpha - b} + k(\alpha - b)\right\} j = 0,$$

which may be integrated to

$$j = \text{const.}\, r^{\frac{1}{2}} \exp\left(\tfrac{1}{2}kr^2\right).$$

Then, by (2.2),

$$\left[\frac{u_1}{c_0} + \frac{\rho_1}{\rho_0}\right]_{b-}^{b+} = \epsilon r^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}kr^2\right), \tag{3.2}$$

where ϵ is a (small) constant. Also, since ψ_α is continuous,

$$\left[\frac{u_1}{c_0} - \frac{\rho_1}{\rho_0}\right]_{b-}^{b+} = 0, \tag{3.3}$$

and since ψ is continuous, (2.15) shows that

$$[v_1]_{b-}^{b+} = 0, \quad [p_1 - c_0^2 \rho_1]_{b-}^{b+} = 0, \quad \left[\frac{c_1}{c_0} - \frac{\gamma - 1}{2}\frac{\rho_1}{\rho_0}\right]_{b-}^{b+} = 0. \tag{3.4}$$

These equations show that the strength of a shock ($\epsilon > 0$) or centred expansion wave ($\epsilon < 0$), as measured by the relative perturbations of the linear theory, decays rapidly when rotation is present ($k > 0$). Note that the Coriolis force, which is represented by the last term of (3.1), played no part in the derivation of this result. The effect is due to the fact that the shock (say) is propagating into a region of increasing pressure and density, so that compressive wavelets are continuously reflected back towards the axis along the other family of characteristics, and this reduces the relative shock strength. These reflexions are not of a conventional type. If we imagine the undisturbed gas to be divided into a large number of annular layers, in each of which the pressure and density are constant, we must also imagine that cylindrical 'membranes', representing the centrifugal force which actually balances the pressure gradient, separate the layers and support the pressure differences. It is difficult to draw conclusions from this model, because the conditions that apply across the membranes in the disturbed state are not obvious, but we may observe that the membranes are fairly 'rigid' in the following sense. Whereas they initially *support* pressure differences according to the law

$$\frac{\delta p_0}{p_0} = 2kr\,\delta r,$$

where δr is the width of an annular layer, they only *transmit* pressure behind a shock according to the law

$$\frac{\delta p_1}{p_1} = kr\,\delta r.$$

(The attenuation represented by the $r^{-\frac{1}{2}}$ in (3.2) has nothing to do with the membranes.)

To obtain a slightly more realistic picture of the propagation of a discontinuity than that given by the simple linearized theory, we apply the method of Whitham (1952). Since no real disturbance can be completely concentrated on the axis, we suppose that the perturbations are due to a cylinder $r = R(t)$ ($R' \ll c_0$). In the linearized solution c_0t is now replaced by $\beta+r$, where β is to be interpreted as constant on a characteristic curve of a higher approximation, and is defined by

$$\frac{1}{c_0}\left(\frac{dr}{dt}\right)_{\beta\,\text{const.}} = 1+\frac{u_1+c_1}{c_0}+\ldots, \quad \beta = c_0t-r \quad \text{on} \quad r = R(t). \tag{3.5}$$

If the equation $r = R(\{\beta+r\}/c_0)$ is written $r = R^*(\beta)$, we then have

$$\left.\begin{aligned} u_1 &= u_1(r,\beta), \quad c_1 = c_1(r,\beta), \\ c_0t-r &= \beta-\int_{R^*(\beta)}^{r}\frac{u_1(y,\beta)+c_1(y,\beta)}{c_0}\,dy. \end{aligned}\right\} \tag{3.6}$$

Two criticisms can be made of this representation in the present context: (i) the α-characteristics and the particle paths now also carry significant disturbances, and (ii) the algebra of the solution (u_1, c_1) is so complicated that (3.6) is not actually usable. Accordingly, we restrict attention to the case of a shock or expansion fan moving into undisturbed fluid; neither objection then is relevant. Let $R(t) = R_0$ for $t \leqslant R_0/c_0$, and let R' be discontinuous at $t = R_0/c_0$ such that

$$R'\left(\frac{R_0+0}{c_0}\right) = \eta.$$

Equations (3.2) to (3.4) apply, with $\beta = b = 0$, and with

$$\epsilon = \frac{2\eta}{c_0}R_0^{\frac{1}{2}}\exp\left(\tfrac{1}{2}kR_0^2\right).$$

By (3.6) the curves $\beta = 0-$ and $\beta = 0+$ become (figure 1)

$$\left.\begin{aligned} \beta = 0-:&\quad c_0t-r = 0, \\ \beta = 0+:&\quad c_0t-r = -\frac{\gamma+1}{4}\epsilon\int_{R_0}^{r}y^{-\frac{1}{2}}\exp\left(-\tfrac{1}{2}ky^2\right)dy, \end{aligned}\right\} \tag{3.7}$$

where the integral can be expressed in terms of the incomplete gamma function. Hence for a shock ($\epsilon > 0$) the regions $\beta < 0$ and $\beta > 0$ overlap in the rt-plane, while for an expansion wave there is a gap between these domains in the rt-plane. The significant point is that, whereas with $k = 0$ the width of these regions is $O(\epsilon r^{\frac{1}{2}})$ for $r \to \infty$, it is now $O(\epsilon)$. Hence in the case of a shock, only a narrow band of characteristics $|\beta| \leqslant O(\epsilon)$ enters the overlapping region in which the shock must lie.

To fit the shock, $t = t_s(r)$ say, we note that to our approximation the shock equations are as follows ([ ] and $\overline{(\,)}$ now denote discontinuities and mean values across the shock):

$$\left[\frac{u_1}{c_0}-\frac{\rho_1}{\rho_0}\right] = 0, \quad [p_1-c_0^2\rho_1] = 0, \quad [v_1] = 0, \tag{3.8}$$

$$c_0\frac{dt_s}{dr} = 1-\frac{\overline{(u_1+c_1)}}{c_0}. \tag{3.9}$$

If k is bounded away from zero, the shock may be drawn halfway between $\beta = 0-$ and $\beta = 0+$, so that

$$c_0t_s = r-\frac{\gamma+1}{8}\epsilon\int_{R_0}^{r}y^{-\frac{1}{2}}\exp\left(-\tfrac{1}{2}ky^2\right)dy, \tag{3.10}$$

and the shock conditions are satisfied to sufficient accuracy. For, by (3·3) and (3.4), the equations (3.8) are satisfied exactly across $\beta = 0$; the shock now lies at $\beta = -|O(\epsilon)|$ and $\beta = |O(\epsilon)|$, and since variations with β are continuous on either side of $\beta = 0$, the equations (3.8) are still satisfied within an error of $O(\epsilon^2)$. Similarly, the characteristic slopes at the shock will be those of $\beta = 0-$ and $\beta = 0+$ within an error of $O(\epsilon^2)$, so that (3.9) is satisfied to sufficient accuracy.

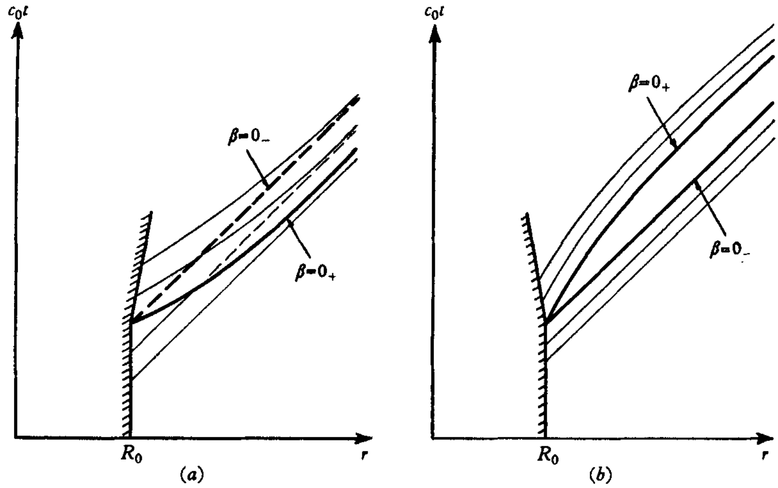

FIGURE 1. (*a*) The 'overlapping region' near a shock; (*b*) the 'gap' produced by a centred expansion wave. Some adjacent characteristics β = constant are also shown qualitatively.

The shock trajectory (3.10) could in fact have been obtained by simply inserting in (3.9) the results of the simple linearized theory for $c_0t-r = 0+$, but the shock would then lie outside the disturbed region of that theory. The advantage of the method used is that it shows how the shock fits into the general flow pattern.

In the case of a centred expansion wave, the gap may be filled by taking a linear variation of u_1 and c_1 across it: this is sufficient to our approximation.

We conclude that, as far as the propagation of discontinuities is concerned, the simple linearized theory (for $k > 0$) is uniformly valid, since the relative perturbations decay rapidly, so that the positions (c_0t-r) of characteristics and shocks are predicted correctly within an error of $O(\epsilon)$ for all r. It would therefore be surprising if the theory, applied to an expanding cylinder of finite radius, contained non-uniformities in the interior part of the field, but discussion of this point is difficult and beyond the scope of the present paper.

4. Waves propagating from the axis

We turn to the main problem of this paper. Consider sources concentrated on the axis: let their strength per unit volume be

$$Q^{(m)} + \frac{\gamma-1}{c_0^2}\,Q^{(h)} = 2\frac{\delta(r)}{r}\,f'(t), \tag{4.1}$$

where δ is the Dirac function, and $f(t) = 0$ for $t < 0$. Integration of (2.11) from $r = 0$ to $r = 0+$ shows that, equivalently, the total source strength per unit length of axis is

$$q(0+, t) = f'(t). \tag{4.2}$$

The resulting flow may be visualized as due to the mass and energy release of (4.1), or as due to a small cylinder of cross-sectional area

$$A(t) \equiv \pi R^2(t) = \frac{2\pi f(t)}{\rho_0(0)}, \tag{4.3}$$

the boundary condition $ru_1 = RR'$ being satisfied on $r = 0$ instead of on $r = R$. We shall examine in detail the solution when the source strength $f'(t)$ is a Dirac pulse (and also one or two closely related cases): in principle, all other solutions can be obtained by superposition of that for a pulse, but in practice this can be a difficult task.

The solutions to be considered are, of course, not physically valid in the immediate neighbourhood of the axis, at or near times when the source is discharging. (In the case of a cylinder, expanding and contracting for $0 \leqslant t \leqslant t_1$, R/c_0t_1, R'/c_0 and $R''t_1/c_0$ must all be $\ll 1$ and continuous if (4.3) is to lead to a physically valid solution in $r \geqslant R$ for all t.) Further, it will be found that if $f(t)$ begins more strongly than $t^{\frac{3}{2}}$ the velocity behind the wave front is infinite. However, these features are also present in the theory for irrotational flow, and therefore do little to hinder the study of rotational effects which is our main purpose here.

In place of r and t we introduce the dimensionless variables

$$x = \frac{\gamma\Omega^2 r^2}{2c_0^2} = kr^2, \quad \tau = \Omega t.$$

That x is in fact a natural co-ordinate of the problem is suggested by the fact that $p_0 = p_0(0)\, e^x$. Then (2.14) becomes

$$x\psi_{xx} - x\psi_x - \frac{1}{2\gamma}\left(\frac{\partial^2}{\partial\tau^2} + 4\right)\psi = 0. \tag{4.4}$$

The boundary conditions, which specify the perturbation mass at the axis and the absence of disturbances ahead of the wave front, are

$$\left.\begin{aligned} &\text{(i)}\ \ \psi(0,\tau) = f(t) = g(\tau), \text{ say,} \\ &\text{(ii)}\ \ \psi = 0 \quad \text{for} \quad r > c_0 t,\ x > \tfrac{1}{2}\gamma\tau^2. \end{aligned}\right\} \tag{4.5}$$

If the Laplace transform

$$\bar{\psi} = p\int_0^\infty e^{-p\tau}\,\psi(x,\tau)\,d\tau$$

is introduced, there results

$$x\bar{\psi}_{xx} - x\bar{\psi}_x - a\bar{\psi} = 0, \quad \text{where} \quad a = \frac{p^2+4}{2\gamma}. \tag{4.6}$$

This is a case of the confluent hypergeometric equation: in the notation of Erdélyi (1953) and Tricomi (1954) the solutions are

$$x\Phi(a+1, 2;\ x) \quad \text{and} \quad \Psi(a, 0;\ x) \equiv x\Psi(a+1, 2;\ x),$$

with the following properties.

For $x \to 0$,
$$\left.\begin{aligned} ax\Phi(a+1,2;x) &= ax + O(x^2), \\ \Gamma(a+1)\Psi(a,0;x) &= 1 + O(x\log x); \end{aligned}\right\} \tag{4.7}$$
for $x \to \infty$,
$$\left.\begin{aligned} ax\Phi(a+1,2;x) &= O(e^x x^a), \\ \Gamma(a+1)\Psi(a,0;x) &= O(x^{-a}); \end{aligned}\right\} \tag{4.8}$$
for $p \to \infty$, $\mathscr{R}p > 0$, $0 < x < \infty$,
$$\left.\begin{aligned} ax\Phi(a+1,2;x) &= \frac{e^{\frac{1}{2}x}}{2\pi^{\frac{1}{2}}}(ax)^{\frac{1}{4}}\exp\{2(ax)^{\frac{1}{2}}\}[1+O(p^{-1})], \\ \Gamma(a+1)\Psi(a,0;x) &= \pi^{\frac{1}{2}}e^{\frac{1}{2}x}(ax)^{\frac{1}{4}}\exp\{-2(ax)^{\frac{1}{2}}\}[1+O(p^{-1})], \\ \text{where}\quad (ax)^{\frac{1}{4}}\exp\{\pm 2(ax)^{\frac{1}{2}}\} &= \left(\frac{\Omega pr}{2c_0}\right)^{\frac{1}{2}}\exp\left(\pm\frac{\Omega pr}{c_0}\right)[1+O(p^{-1})]. \end{aligned}\right\} \tag{4.9}$$

Here $\arg a = \arg p = 0$ on the positive real p-axis. Equation (4.8) suggests that the Φ-solution should be rejected here, and (4.9) confirms it. For, in view of the inversion integral of the Laplace transform, (4.9) shows that the Φ-solution represents an incoming wave, which vanishes only for $c_0t+r < 0$; whereas Ψ represents an outgoing wave which vanishes for $c_0t-r < 0$, as required by (4.5) (ii). Hence
$$\bar{\psi} = \Gamma(a+1)\Psi(a,0;x)\bar{g}(p),$$
and
$$\psi = \frac{1}{2\pi i}\int_{\mathscr{L}}\frac{e^{p\tau}}{p}\Gamma(a+1)\Psi(a,0;x)\bar{g}(p)\,dp, \tag{4.10}$$
where $\mathscr{L}$ is the Bromwich path $C-i\infty$ to $C+i\infty$ with $C > 0$.

To relate this result to those of the previous section, we suppose that
$$f(t) \sim t^\nu H(t)/\nu! \quad \text{for} \quad t \to 0 \quad (\nu \geqslant 0),$$
where $H(t)$ is the Heaviside function. (In applications such an $f(t)$ must, of course, be multiplied by a small parameter ϵ.) Applying (4.9) and $\bar{g}(p) \sim (\Omega p)^{-\nu}$ for $p \to \infty$, and closing the contour of (4.10) with a large semicircle in $\mathscr{R}p > 0$, we confirm that $\psi = 0$ for $r > c_0t$; evaluating the integral for $r < c_0t$ we find that immediately behind the wave front
$$\psi \sim \frac{\pi^{\frac{1}{2}}}{(\nu-\frac{1}{2})!}e^{\frac{1}{2}x}\left(\frac{r}{2c_0}\right)^{\frac{1}{2}}\left(t-\frac{r}{c_0}\right)^{\nu-\frac{1}{2}}. \tag{4.11}$$
The case studied in §3, in which ψ_r and ψ_t have finite discontinuities, is $\nu = \frac{3}{2}$; if $\nu < \frac{3}{2}$ the theory is unrealistic near the wave front, and if $\nu > \frac{3}{2}$ the strength of the wave front is zero. The effects of rotation, however, are contained entirely in the factor $e^{\frac{1}{2}x}$, which becomes $e^{-\frac{1}{2}x}$, in the relative perturbations, and has already been discussed.

Consider now the solution
$$\psi_{(0)} = \frac{1}{2\pi i}\int_{\mathscr{L}}\frac{e^{p\tau}}{p}\Gamma(a+1)\Psi(a,0;x)\,dp, \tag{4.12}$$
for which $\psi_{(0)}(0,\tau) = H(\tau)$, $q_{(0)}(0,\tau) = \Omega\delta(\Omega t) = \delta(t)$, so that the source strength is a Dirac pulse. For any other source strength
$$\psi = \int_0^{\tau-(2x/\gamma)^{\frac{1}{2}}}\psi_{(0)}(x,\tau-\tau_1)\,g'(\tau_1)\,d\tau_1.$$

First we verify that when $x \to 0$ and $\tau \to 0$ such that x/τ^2 is finite, the irrotational solution is recovered. Let

$$\frac{r^2}{c_0^2 t^2} = \frac{2x}{\gamma\tau^2} = \theta^2, \quad p\tau = s,$$

so that $$\psi_{(0)} = \frac{1}{2\pi i}\int_{\mathscr{L}} \frac{e^s}{s}\Gamma(a+1)\Psi(a, 0; \tfrac{1}{2}\gamma\tau^2\theta^2)\,ds, \quad a = \frac{s^2+4\tau^2}{2\gamma\tau^2}.$$

Now for $\tau \to 0$ (Erdélyi, 1953, p. 266)

$$\Gamma\Psi = s\theta K_1(s\theta)\,[1+O(\tau^2)],$$

where K_1 is the modified Bessel function of the second kind and of order one. Hence

$$\begin{aligned}\psi_{(0)} &= \frac{1}{2\pi i}\int_{\mathscr{L}} e^s\theta K_1(s\theta)\,ds\,[1+O(\tau^2)] \\ &= (1-\theta^2)^{-\frac{1}{2}}[1+O(\tau^2)],\end{aligned} \tag{4.13}$$

and this is the irrotational solution.

A wave-front approximation which, while in principle no better than (4.11) (with $\nu = 0$), actually gives better numerical agreement with the full solution, may be obtained as follows. We observe that the Coriolis force contributed the stiffness term $-4\Omega^2\psi/c_0^2$ in (2.14), or $-2\psi/\gamma$ in (4.4), and that this is lost when the approximation $a \sim p^2/2\gamma$ for $p \to \infty$ is made in (4.9). Accordingly, this approximation is not made: then

$$\psi_{(0)} \sim \frac{\pi^{\frac{1}{2}}e^{\frac{1}{2}x}}{2\pi i}\int_{\mathscr{L}} \exp\{p\tau - 2(ax)^{\frac{1}{2}}\}(ax)^{\frac{1}{4}}\frac{dp}{p} \quad \left(a = \frac{p^2+4}{2\gamma}\right),$$

and the integral may be evaluated approximately by the methods of steepest descent or of stationary phase. There results

$$\begin{aligned}\psi_{(0)} &\sim e^{\frac{1}{2}x}\frac{2x}{\gamma\tau^2}\left(1-\frac{2x}{\gamma\tau^2}\right)^{-\frac{1}{2}}\cos\left\{2\left(\tau^2-\frac{2x}{\gamma}\right)^{\frac{1}{2}}\right\} \\ &\sim e^{\frac{1}{2}x}\left(1-\frac{2x}{\gamma\tau^2}\right)^{-\frac{1}{2}}\cos\left\{2\left(\tau^2-\frac{2x}{\gamma}\right)^{\frac{1}{2}}\right\}.\end{aligned} \tag{4.14}$$

This is still only valid for small values of $(\gamma\tau^2 - 2x)$, but it appears to hold over a somewhat wider range than (4.11), and it also gives a qualitative indication of the oscillatory character of the flow for $\Omega > 0$.

To obtain a series representation of $\psi_{(0)}$, we note that in (4.12) the only singularities of the integrand are simple poles at the origin and at those points on the imaginary p-axis where a is a negative integer. For $p \to \infty$ and $\mathscr{R}p < 0$,

$$\Gamma(a+1)\Psi(a, 0; x) \sim O[a^{\frac{1}{4}}\exp\{2(ax)^{\frac{1}{2}}\}],$$

so that the integral around a large semicircle in $\mathscr{R}p < 0$ vanishes for $c_0 t > -r$. Evaluating the residues, one finds that

$$\psi_{(0)} = \Gamma\left(\frac{2}{\gamma}+1\right)\Psi\left(\frac{2}{\gamma}, 0; x\right) - x\sum_{n=1}^{\infty}\frac{\cos\{(2\gamma n+4)^{\frac{1}{2}}\tau\}L_{n-1}^1(x)}{n+(2/\gamma)} \quad (x > 0), \tag{4.15}$$

where $L^1_{n-1}(x)$ is the generalized Laguerre polynomial

$$L^1_{n-1}(x) = \frac{n}{1!} - \frac{n(n-1)}{2!}\frac{x}{1!} + \ldots + (-)^{n-1}\frac{x^{n-1}}{(n-1)!},$$

with the recurrence property

$$L^1_m(x) = \left(2 - \frac{x}{m}\right) L^1_{m-1}(x) - L^1_{m-2}(x).$$

For $n \to \infty$,

$$\frac{\cos\{(2\gamma n+4)^{\frac{1}{2}}\tau\} L^1_{n-1}(x)}{n+(2/\gamma)} \sim -\frac{e^{\frac{1}{2}x}}{2\pi^{\frac{1}{2}}(nx)^{\frac{3}{4}}}\left[\cos\left\{(2\gamma n)^{\frac{1}{2}}\,\Omega\left(t-\frac{r}{c_0}\right)-\tfrac{1}{4}\pi\right\} + \cos\left\{(2\gamma n)^{\frac{1}{2}}\,\Omega\left(t+\frac{r}{c_0}\right)+\tfrac{1}{4}\pi\right\}\right]. \quad (4.16)$$

In (4.15) the first term is the solution obtained if in the differential equation (2.14) one lets $\Omega \to \infty$, $c_0 \to \infty$ with Ω/c_0 finite, and applies the boundary conditions $\psi = 1$ on $x = 0$, $\psi = o(e^x)$ for $x \to \infty$. Since $\psi = 1$ on $x = 0$ implies $t > 0$, this limit corresponds to $\tau \to \infty$ with x fixed; it therefore seems probable (although it has not been proved) that the second part of the solution $\to 0$ as $\tau \to \infty$, and in this case the first term can be regarded as the ultimate steady-state form of the solution. It can be shown that the series in (4.15) converges for $c_0 t > r$, although not absolutely; and that if we sum and then let $x \to 0$ the second part of the solution is $O(x)$. This last remark means that in the perturbation density, $\rho_1 = -\gamma\Omega^2 c_0^{-2}\psi_x$, the dominant term for $x \to 0$ is the logarithmic singularity implied by

$$\Gamma\left(\frac{2}{\gamma}+1\right)\Psi\left(\frac{2}{\gamma},0;x\right) = 1 + \frac{2x}{\gamma}\log\frac{2x}{\gamma} + O(x).$$

The pressure is singular in the same way. This singularity (which appears for all source strengths), is due to the arrival at the axis of the wavelets reflected by the radial pressure gradient. Note that in (4.16) the high-frequency components of the solution are in fact split into outgoing and incoming waves.

5. An example

Consider the family of solutions $\psi_{(m)}$ (m an integer $\geqslant 0$) for which

$$\psi_{(m)}(0,\tau) = \tau^m H(\tau).$$

For $m = 0$ the wave-front singularity is strong, and the series in (4.15) converges very slowly throughout the field: corresponding series solutions can be written for all m, and the convergence improves as m increases, but for m large the growth of outflow with time is likely to be the dominant feature of the field. Hence $m = 1$ was chosen here; and for convenience γ was taken to be 2. The equations which correspond to (4.14) and (4.15) are then

$$\psi_{(1)} = \tfrac{1}{2}e^{\frac{1}{2}x}\sin\{2(\tau^2-x)^{\frac{1}{2}}\} \quad (\tau^2 - x \text{ small}), \quad (5.1)$$

and

$$\psi_{(1)} = \tau\{1 + xe^x\mathrm{Ei}(-x)\} - \tfrac{1}{2}x\sum_{n=1}^{\infty}\frac{\sin\{2(n+1)^{\frac{1}{2}}\tau\}L^1_{n-1}(x)}{(n+1)^{\frac{3}{2}}}, \quad (5.2)$$

where Ei denotes the exponential integral.

The series was computed as follows. A large value $n = N$ was chosen, close to a zero of $\sin\{2(n+1)^{\frac{1}{2}}\tau\}$ for computations with τ constant, and close to a zero of $L^1_{n-1}(x)$ for computations with x constant. For $n \leqslant N$ the terms were computed exactly, the recurrence formula of the Laguerre polynomials being used. For $n > N$ the asymptotic forms corresponding to (4.16) were used, and to the same degree of approximation, summation could be replaced by integration. The sum for $n > N$ could then be expressed in terms of tabulated Fresnel integrals (Jahnke & Emde, 1945). Two values of N were used for each point, one near 20 and the other near 40, and the difference $\Delta\psi_{(1)}/\psi_{(1)}(0,\tau)$ was less than 2 % in all cases.

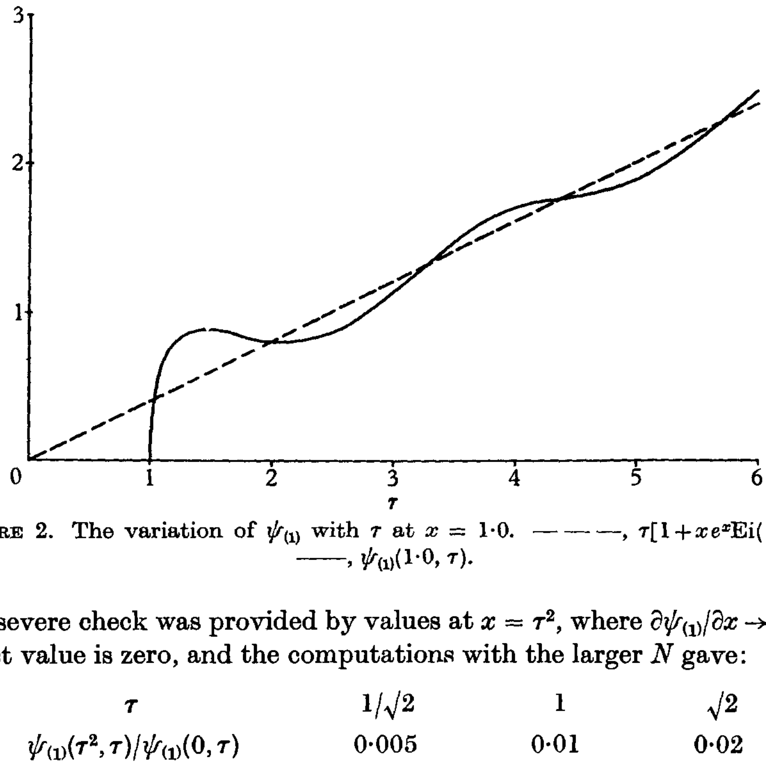

FIGURE 2. The variation of $\psi_{(1)}$ with τ at $x = 1{\cdot}0$. — — —, $\tau[1+xe^x\mathrm{Ei}(-x)]$; ——, $\psi_{(1)}(1{\cdot}0, \tau)$.

A more severe check was provided by values at $x = \tau^2$, where $\partial\psi_{(1)}/\partial x \to \infty$; here the exact value is zero, and the computations with the larger N gave:

τ	$1/\sqrt{2}$	1	$\sqrt{2}$
$\psi_{(1)}(\tau^2,\tau)/\psi_{(1)}(0,\tau)$	0·005	0·01	0·02

Because $\psi_{(m)}(x,\tau) = \partial\psi_{(m+1)}(x,\tau)/\partial\tau$, the function $\epsilon\psi_{(1)}$ can be interpreted:

(i) As a function whose time derivative (indicated by figure 2) is proportional to the solution $\psi_{(0)}$ of the previous section.

(ii) As the stream function ψ of the flow for which

$$f(t) = \epsilon\Omega t, \quad R(t) = \{2\epsilon\Omega/\rho_0(0)\}^{\frac{1}{2}} t^{\frac{1}{2}}.$$

(iii) As the outflow q of the flow for which

$$f(t) = \tfrac{1}{2}\epsilon\Omega t^2, \quad R(t) = \{\epsilon\Omega/\rho_0(0)\}^{\frac{1}{2}} t.$$

Let us adopt the second of these viewpoints and assume that $\epsilon > 0$. Figure 2 shows the variation of $\psi_{(1)}$ with time at $x = 1$, and we take $\psi = r\rho_0\int_{r/c_0}^{t} u_1\,dt$ as a measure of the radial displacement of the circle of fluid particles which lies initially at $x = 1$. As the shock passes by, the circle surges outwards behind it (at first with infinite velocity), but this effect is soon checked by the spring-like Coriolis force, and also by a fall in density which makes the pressure gradient p_{0r} more effective, so that the circle moves inwards again. (According to the wave-front approximation (5.1), ρ_1 first becomes negative at $x = 1{\cdot}0$ when $\tau = 1{\cdot}2$.) Thereafter the particles' mean motion is outward, because of the continuing efflux from the source, but they oscillate about the mean trajectory shown by the straight line in figure 2. This line is the 'infinite rotation, infinite sound speed' solution given by the first term of (5.2).

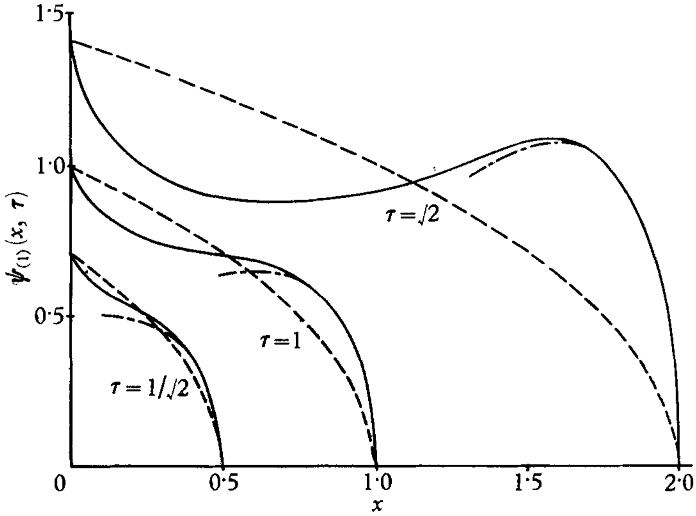

FIGURE 3. Profiles of $\psi_{(1)}$ at constant τ. ——, full solution, equation (5.2); — - — -, wave-front approximation, equation (5.1); — — —, irrotational-flow solution, equation (5.3).

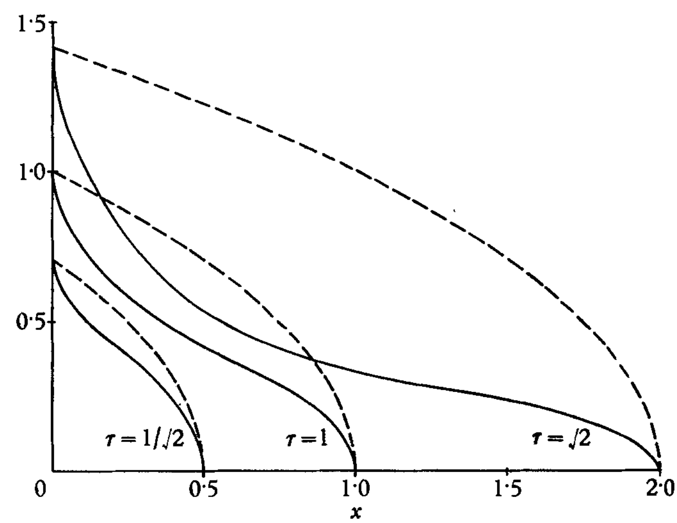

FIGURE 4. Profiles of $e^{-x}\psi_{(1)}$ at constant τ. ——, $e^{-x}\psi_{(1)}(x, \tau)$; — — —, irrotational-flow solution, equation (5.3).

Figures 3 and 4 show profiles of $\psi_{(1)}$ and $e^{-x}\psi_{(1)}$ at constant τ; the wave-front approximation (5.1) is shown also, and the shape of the irrotational solution

$$\Omega\psi^*_{(1)} = (\tau^2 - x)^{\frac{1}{2}} \tag{5.3}$$

serves as a datum. Since

$$\rho_1 = -\frac{\gamma\Omega^2}{c_0^2}\psi_x,$$

the slope of the curves in figure 3 indicates the (absolute) density perturbation: the figure also shows the effect of rotation in gradually distorting the $\psi_{(1)}$-profile from its shape in the irrotational case. The most striking feature is perhaps the appearance at $\tau = \sqrt{2}$ of a wide region of negative perturbation density (rarefaction) between the regions of strongly positive ρ_1 near the axis and the wave front. Of course, the presence of these strong condensation regions (which we attribute to the convergence on the axis of the reflected wavelets, and to the negative damping of the absolute density increment behind the wave front) makes the appearance of an intermediate rarefaction region inevitable.

The relative perturbation $e^{-x}\psi_{(1)}$ of figure 4 is a measure, not only of the area displaced by a fluid circle, but also of the loss of circulation and gain of entropy at a station x = constant. For the circles of fluid particles carry their initial circulation and entropy with them, and

$$rv_1 = -\frac{2\Omega\psi}{\rho_0}, \quad S_1 = \frac{c_v}{p_0}(p_1 - c_0^2\rho_1) = \frac{c_v(\gamma-1)\,\Omega^2}{p_0}\psi,$$

where S is the specific entropy. (Alternatively, $e^{-x}\psi_{(1)}$ may be associated with the volumetric outflow ru_1 of case (iii) above.) The figure emphasizes the resistance to small disturbances of the rotating mass of gas: the relative perturbations decay rapidly as one proceeds outwards from the axis.

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